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LOCALLY CONNECTED SIMPLICIAL MAPS*

BY

M. ASCHBACHER

Department of Mathematics
California Institute of Technology, Pasadena, CA 91125, USA

AND

Y. SEGEV

Department of Mathematics and Computer Science
Ben Gurion University of the Negev, Beer Sheva 84105, Israel

ABSTRACT

In Propositions 1.6 and 7.6 of his paper on p-group complexes of finite groups [5], Quillen establishes fundamental results comparing the homology and the fundamental group of the order complexes of posets P, Q admitting a map $f : P \to Q$ of posets with good local behavior. We prove the analogue of Quillen's results for maps $f : K \to L$ of simplicial complexes K and L in a more general setup.

Let K be a finite dimensional simplicial complex with geometric realization $|K|$. (See [3], [6], and the body of the paper for definitions of standard terminology.) We say that K is **n-connected** if $|K|$ is n-connected as a topological space and K is **contractible** if $|K|$ is contractible.

Define a simplicial map $\phi : L \to D$ of finite dimensional simplicial complexes to be **locally n-connected** if for all $0 \leq k \leq n+1$ and all k-dimensional simplices s of D, $\bigcap_{a\in s} \phi^{-1}(st_D(a))$ is $(n-k)$-connected. Here $\phi^{-1}(st_D(a))$ is the subcomplex of L consisting of all simplices t with $\phi(t)$ a simplex of $st_D(a)$. Term ϕ **locally contractible** if ϕ is locally n-connected for all n; that is $\bigcap_{x\in s} \phi^{-1}(st_D(x))$ is contractible for all simplices s of D.

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The **graph** of D is the (undirected) graph $\Delta(D)$ on the vertices of D in which vertices are adjacent if they form a simplex of D. Conversely the **clique complex** of a graph Δ is the simplicial complex $K(\Delta)$ with vertex set Δ and simplices the cliques of Δ. We say D is a **clique complex** if $D = K(\Delta(D))$. For example the order complex of a poset is a clique complex and more generally D is a clique complex if and only if $st_D(s) = \bigcap_{a\in s} st_D(a)$ for each simplex s of D. (**cf.** 2.1)

The following result should be compared to Propositions 1.6 and 7.6 in Quillen [5]:

THEOREM 1: *Assume $\phi : L \to D$ is a locally n-connected simplicial map of finite dimensional simplicial complexes with D a clique complex. Then*

1. *$|\phi| : |L^n| \to |D^n|$ is a homotopy equivalence of n-skeletons.*
2. *If $n \geq 1$ then $\phi_* : \pi_1(|L|, x) \to \pi_1(|D|, \phi(x))$ is an isomorphism for each vertex x of L.*
3. *L is n-connected if and only if D is n-connected.*
4. *If ϕ is locally contractible then $|\phi| : |L| \to |D|$ is a homotopy equivalence.*

We have stronger results on simple connectivity. Define an **n-approximation** of a simplicial complex L by a simplicial complex D to be a surjection $\theta : \Delta(D) \to \mathcal{F}$ of $\Delta(D)$ onto some family $\mathcal{F} = \mathcal{F}(\theta)$ of subcomplexes of L such that each k-simplex of L is contained in a member of $\mathcal{F}$, for all $0 \leq k \leq n$, and

(n-Approx) $\bigcap_{a\in s} \theta(a)$ is an $(n-k)$-connected subcomplex of L for each $0 \leq k \leq n+1$ and each k-simplex s of D.

THEOREM 2: *Let $\phi : L \to D$ be a locally 0-connected simplicial map and x a vertex of L. Then the induced map $\phi_* : \pi_1(|L|, x) \to \pi_1(|D|, \phi(x))$ is a surjection, so in particular if L is simply connected then so is D.*

THEOREM 3: *Let L, D be simplicial complexes with graphs Λ, Δ, respectively, and assume θ is a 1-approximation of L by D. For $x \in \Lambda$ let $\mathcal{F}(x) = \{a \in \Delta : x \in \theta(a)\}$ regarded as a subgraph of Δ. Assume that whenever $a, b \in \Delta$ and $x \in \theta(a) \cap \theta(b)$ that there exists $y \in \theta(a) \cap \theta(b)$ such that a and b are in the same connected component of $\mathcal{F}(y)$ and x and y are in the same connected component of $\theta(a) \cap \theta(b)$. Then if D is simply connected, so is L.*

We also establish a variety of more specialized results. See in particular lemma 3.3 that supplies a sufficient condition for nonvanishing of 1-homology and Theorem 2.5 which is a version of Theorem 1 in the case $n = 1$ which does not require D to be a clique complex.

Our results will be applied in later papers to certain simplicial complexes associated to finite groups. For example in [1,2] we began a program to establish the uniqueness of the sporadic finite simple groups via an analysis of the simple connectivity of certain graphs and geometries associated to these groups. In addition our results can be used to study the Brown and Quillen complexes of finite groups at the prime p from the point of view of the clique complex on the commuting graph of subgroups of order p. (cf. [8]) For while Propositions 1.6 and 7.6 in [5] are well suited for studying the order complex of a poset, Theorem 1 is better suited to the study of the clique complex of a graph.

1. Generalities: simple connectivity

In this section we record some of the basic facts about simplicial complexes we will need. Some are standard facts from combinatorial topology and others extend the point of view of [1] to simplicial complexes. The reader is referred to [1] for our graph theoretic notation and terminology.

Recall the **star** $st_D(s)$ of a simplex s of a simplicial complex D is the subcomplex of D whose simplices are the simplices t of D such that $s \cup t$ is a simplex of D.

Define a simplicial map $\phi : L \to D$ of simplicial complexes to be a **covering** if ϕ is surjective on vertices and a local isomorphism; that is for each vertex x of L, the map $\phi_x : st_L(x) \to st_D(\phi(x))$ is an isomorphism.

Write L^n for the n-skeleton of L; notice we can identify L^1 with the graph $\Delta(L)$ of L from the introduction. Notice also that if $\phi : L \to D$ is a simplicial map then the induced map $\phi : \Delta(L) \to \Delta(D)$ is a morphism of graphs, while conversely each morphism $d : \Lambda \to \Delta$ of graphs induces a simplicial map $d : K(\Lambda) \to K(\Delta)$ of clique complexes. Moreover

(1.1): *If $\phi : L \to D$ is a covering of simplicial complexes then*

(1) *ϕ is surjective on simplices, and*

(2) *For each simplex s of L, $\phi : st_L(s) \to st_D(\phi(s))$ is an isomorphism.*

Let $P = P(\Delta)$ be the set of paths in the graph Δ of some simplicial complex D. Let $\sim$ be a P-invariant equivalence relation on P as defined in section 2 of [1]. As in [1], define $P/\sim = \tilde{P}$ to be the set of equivalence classes of $\sim$ and make $\tilde{P}$ into a graph as in section 4 of [1].

Notice that $\tilde{P}$ is a groupoid; ie. a small category in which all morphisms are isomorphisms. Namely Δ is the collection of objects of this category while the morphisms from x to y are the classes $\tilde{p}$ with p a path from x to y. We abuse terminology and call P a groupoid too, although here each morphism is not an isomorphism. But at least we have the contravariant functor $p \mapsto p^{-1}$ (trivial on Δ), which is almost as good.

Pick a vertex $x \in \Delta$. As in section 4 of [1], let $\Sigma(\Delta, x)$ denote the set of paths with origin x, write $\tilde{\Sigma}(\Delta, x) = \Sigma(\Delta, x)/\sim$ for the set of classes $\tilde{p}$ with $p \in \Sigma(\Delta, x)$, and $\tilde{\pi}(\Delta, x)$ for the group of $\tilde{p} \in \tilde{\Sigma}(\Delta, x)$ with p a cycle. From 4.1 in [1] we have a map end : $\tilde{p} \mapsto \mathrm{end}(p)$ from $\tilde{\Sigma}(\Delta, x)$ into Δ.

Define $\tilde{D}$ to be the subcomplex of $K(\tilde{P})$ with simplices s such that $\mathrm{end}(s) = \bigcup_{a \in s} \mathrm{end}(a)$ is a simplex of D. Let $\tilde{\Sigma}(D, x)$ be the corresponding subcomplex of $\tilde{\Sigma}(\Delta, x)$ and $\tilde{\pi}(D, x)$ the corresponding group. Let $\pi_1(D, x) = \tilde{\pi}(D, x)$, where $\sim = \sim_S$ for S the closure (**cf.** section 2 in [1]) of the set of all 2-simplices of D. Then $\pi_1(D, x)$ is the **fundamental group** of D and is isomorphic to the fundamental group of the geometric realization of D:

(1.2): $\pi_1(D, x_0) \cong \pi_1(|D|, x_0)$.

Proof: This is a standard result stated in a slightly different language. Namely $\pi_1(D, x_0)$ is the edge path group of D.(**cf.** Chapter 3, Section 6 of [6]) Now appeal to Theorem 3.6.16 in [6]. ∎

(1.3): *Assume D is a simplicial complex with connected graph Δ, and $\sim$ is a P-invariant equivalence relation on P. Then* end : $\tilde{D} \to D$ *is a covering of simplicial complexes if and only if* $\ker(\sim)$ *contains all 2-simplices of D.*

Proof: See 4.1 in [1]. The **kernel** $ker(\sim)$ of $\sim$ is defined in section 2 of [1]. ∎

Define a **local system** on the graph Δ to be an assignment $F : x \mapsto F(x)$, $(x, y) \mapsto F_{xy}$ of each $x \in \Delta$ to a set $F(x)$ and each edge (x, y) to a bijection $F_{xy} : F(x) \to F(y)$, such that

$$(LS1) \qquad F_{yx} \circ F_{xy} = id_{F(x)} \text{ for each edge } (x, y) \text{ of } \Delta.$$

Example 1: If $d : \Lambda \to \Delta$ is a fibering of graphs in the sense of [1], we obtain a local system F^d on Δ defined by $F^d(x) = d^{-1}(x)$ and $F^d_{xy}(\alpha) = d_\alpha^{-1}(y)$, where $d_\alpha = d_{|\Lambda(\alpha)}$.

Given a local system F on Δ, we can proceed as in section 4 of [1]. Namely if $p = x_0 \cdots x_n \in P(\Delta)$ is a path, define $F_p : F(org(p)) \to F(end(p))$ recursively by $F_{x_0} = id_{F(x_0)}$ and $F_p = F_{x_{n-1}x_n} \circ F_q$, where $q = x_0 \cdots x_{n-1}$. We say paths p and q are **F-homotopic** if $F_p = F_q$ and write $p \sim_F q$.

(1.4): *(1) If p, q are paths with $end(p) = org(q)$ then $F_{pq} = F_q \circ F_p$.*

(2) $F_p \circ F_{p^{-1}} = id_{F(end(p))}$.

(3) F-homotopy is a $P(\Delta)$-invariant equivalence relation.

(4) F is a functor from the path groupoid of Δ to Sets, so that F_p is an isomorphism for each path p.

Let D be a simplicial complex with graph Δ. A **local system** on D is a local system on Δ such that:

(LS2) $F_{xz} = F_{yz} \circ F_{xy}$ for each 2-simplex $\{x, y, z\}$ of D.

Example 2: If $d : L \to D$ is a covering of simplicial complexes then d induces a fibering $d : \Lambda \to \Delta$ of the graphs of L, D and hence a local system F^d on Δ by Example 1. Then F^d is a local system on D.

We can view D as a category whose objects are the simplices of D and morphisms are inclusions of simplices.

(1.5): *If F is a local system on D then F induces a functor F from D to Sets via $F(s) = F(x_s)$ for some $x_s \in s$ and $F_{s,t} = F_p$ for some path p from x_s to x_t in t.*

Notice the definition in 1.5 is independent of p as t is simply connected. Lemmas 1.4.4 and 1.5 show that a local system on Δ or D is a local system in the sense of section 7 of [5].

Given a local system F on Δ, define Δ_F to be the disjoint union of the sets $F(x)$, $x \in \Delta$, and define $d_F : \Delta_F \to \Delta$ to be the map with $d_F^{-1}(x) = F(x)$ for each $x \in \Delta$. Make Δ_F into a graph by decreeing that u is adjacent to v if there is an edge (x, y) of Δ with $u \in F(x)$ and $F_{xy}(u) = v$. Notice (LS1) says this relation is symmetric.

Similarly if F is a local system of D let D_F be the subcomplex of $K(\Delta_F)$ with simplices s such that $d_F(s)$ is a simplex of D.

If Γ is a graph and v a vertex of Γ, set $v^\perp = \{v\} \cup \Gamma(v)$.

(1.6): *Let F be a local system on D. Then*

(1) $d_F : D_F \to D$ is a covering of D.

(2) $F = F^{d_F}$ is the local system of d_F.

(3) If $\phi : L \to D$ is a covering then $D_{F^\phi} = L$ and $\phi = d_{F^\phi}$.

Proof: Let $d = d_F$, $x \in \Delta$, and $u \in F(x)$. Then $u^\perp = \{F_{xy}(u) : y \in x^\perp\}$ and the map $F_{xy}(u) \mapsto y$ is the restriction d_u of d to $u^\perp$. In particular $d_u : u^\perp \to x^\perp$ is a bijection so d is a fibering. Further if $t = \{x, y, z\}$ is a 2-simplex in D, then $F_{xz}(u) = F_{yz}(F_{xy}(u)) = F_{xyz}(u)$, so $F_{xz}(u)$ is adjacent to $F_{xy}(u)$ in Δ_F and hence $d_u^{-1}(xyzx)$ is a triangle in Δ_F. So $d_u^{-1}(t)$ is a 2-simplex of D_F and (1) holds. Parts (2) and (3) follow from the definitions. ∎

(1.7): *The map $F \mapsto d_F$ is a natural equivalence of local systems on D with coverings of D. The inverse of this equivalence is $d \mapsto F^d$.*

Proof: This follows from 1.5. ∎

Given a local system on D let $\lim_{x\in\Delta}(F(x))$ be the disjoint union of the $F(x)$ modulo the equivalence relation $\sim$ generated by the identifications $F_{xy} : F(x) \to F(y)$, (x, y) an edge of Δ. Observe:

(1.8): *Let F be a local system. Then*

(1) For $u, v \in \coprod_{x\in\Delta}(F(x))$, $u \sim v$ in $\lim_{x\in\Delta}(F(x))$ if and only if there exists a path p from $x = d_F(u)$ to $y = d_F(v)$ with $F_p(u) = v$.

(2) If D is simply connected then the map $u \mapsto \bar{u}$ is a bijection of $F(x)$ with $\lim_{x\in\Delta}(F(x))$, where $\bar{u}$ is the equivalence class of u under $\sim$.

Proof: Part (1) follows from 1.4. Assume D is simply connected. By 1.6, d_F is a covering of D with local system F. As D is simply connected, all cycles are trivial with respect to the local system of d_F. Thus by (1) the map $\alpha : u \mapsto \bar{u}$ is an injection of $F(x)$ into $L = \lim_{x\in\Delta}(F(x))$. As Δ is connected, (1) also says this map is a surjection, so (2) holds.

(1.9): *Let $d : L \to D$ be a connected covering of complexes, Δ the graph of D, $F = F^d$, and $\sim$ the relation on $P = P(\Delta)$ defined by $p \sim q$ if $F_p = F_q$. Then*

(1) $\sim$ is a P-invariant relation.

(2) Let α be a vertex of L and $x = d(\alpha)$. Then the map

$$\psi : \tilde{\Sigma}(D, x) \to L$$

defined by $\psi(\tilde{p}) = F_p(\alpha)$ is a covering with $d \circ \psi = end$.

(3) For $p \in P$ with $org(p) = x$ there exists a unique path q in L with $org(q) = \alpha$ and $d(q) = p$. Further $F_p(\alpha) = end(q)$.

Proof: See 4.2 in [1] for (1) and (2). Part (3) follows by induction on the length of p. ∎

(1.10): *Let D be a connected simplicial complex, S the closure of all 2-simplices of D, $\sim = \sim_S$, and $x \in \Delta$. Then*

(1) $end : \tilde{\Sigma}(D, x) \to D$ is a universal covering of D.

(2) For p a cycle at x, $1 \neq \tilde{p}$ in $\pi_1(D, x)$ if and only if $F_p \neq id$, where F is the local system for end.

Proof: Part (1) follows from 1.9.2 and 1.3. Let $p = x_0 \cdots x_r$ and q be cycles at x, $p_i = x_0 \cdots x_i$, and $q_i = qp_i$. Then by 1.9.3, $\tilde{q}_0 \cdots \tilde{q}_r$ is the unique lift of p under end with origin $\tilde{q}$ and $F_p(\tilde{q}) = \tilde{q}_r = \widetilde{qp} = \tilde{q} \cdot \tilde{p}$. Thus $F_p = id$ if and only if $\tilde{p} = 1$. ∎

(1.11): *For a connected simplicial complex D, the following are equivalent:*

(1) D possesses no nontrivial connected coverings.

(2) $P(\Delta)$ is generated by the 2-simplices of D.

(3) $F_p = id$ for each local system F on D and each cycle p of Δ.

(4) $\pi_1(D) = 0$.

Proof: Parts (1) and (2) are equivalent by 1.10.1. Indeed $\pi_1(D, x)$ is the fiber of $x \in D$ under the universal covering *end* of 1.10.1, so (1) and (4) are equivalent. Finally (3) and (4) are equivalent by 1.10.2. ∎

In particular 1.2 and 1.11 say:

(1.12): *D is simply connected if and only if D possesses no nontrivial connected covering.*

2. The proofs of Theorems 2 and 3

In this section L, D are simplicial complexes and $\phi : L \to D$ is a simplicial map. Let Λ and Δ be the graphs of L and D, respectively, so that the map $\phi : \Lambda \to \Delta$ of vertices induced by ϕ is a morphism of graphs.

Recall that a topological space T is n-**connected** if $T \neq \emptyset$ and for all $0 \leq k \leq n$, each continuous map from S^k to T extends to a continuous map from E^{k+1} to T. Recall also that the space $|K|$ is n-connected if and only if $\tilde{H}_k(|K|) = 0$ for

$-1 \leq k \leq n$ and $|K|$ is simply connected if $n \geq 1$. Here $\tilde{H}_k(|K|)$ is the kth reduced homology group of $|K|$. (*cf.* [6]) Notice a space is -1-connected if and only if it is nonempty.

Recall from the introduction the definitions of n-connectivity of a simplicial complex and local n-connectivity of a simplicial map. We say ϕ is **locally connected** if ϕ is locally 0-connected and ϕ is **locally simply connected** if ϕ is locally 1-connected.

Recall a **cover** of a simplicial complex D is a collection $\mathcal{F}$ of subcomplexes of D such that each simplex of D is contained in some member of $\mathcal{F}$. The **nerve** $N(\mathcal{F})$ of the cover is the simplicial complex whose simplices are the subsets S of $\mathcal{F}$ such that $\bigcap_{S \in S} S \neq \emptyset$.

(2.1): *Let D be a simplicial complex; then the following are equivalent:*

(1) *D is a clique complex.*

(2) *For each simplex s of D, $st_D(s) = \bigcap_{x \in s} st_D(x)$.*

(3) *For each 1-simplex $\{x, y\}$ of D, $st_D(x) \cap st_D(y) = st_D(\{x, y\})$.*

Proof: It is easy to see that (1) implies (2) and of course (2) implies (3). Assume (3); we prove each clique $c = \{x_1, \ldots x_n\}$ of Δ is a simplex of D by induction on n. By definition of Δ, we may take $n > 2$, and by induction, $\{x_1\} \cup s$ and $\{x_2\} \cup s$ are simplices, where $s = \{x_3, \ldots x_n\}$. Thus s is a simplex of $st_D(x_1) \cap st_D(x_2) = st_D(\{x_1, x_2\})$, so $c = \{x_1, x_2\} \cup s$ is a simplex. ∎

(2.2): *Let $v \in \Lambda$, and assume $\phi : L \to D$ is locally connected. Then ϕ induces a surjective group homomorphisms $\phi : \pi_1(L, v) \to \pi_1(D, \phi(v))$ via $\phi(\tilde{p}) = \widetilde{\phi(p)}$ for p a cycle at v.*

Proof: Let $x = \phi(v)$. Write $\sim$ for the relation $\sim_S$ on Λ and Δ, where S is the closure of the set of 2-simplices of the respective complexes in the respective graphs. Define $\phi : \tilde{P}(\Lambda) \to \tilde{P}(\Delta)$ by $\phi(\tilde{p}) = \widetilde{\phi(p)}$. As $\phi(ker(\sim)) \leq ker(\sim)$, (*cf.* 1.3 in [2]) the map is well defined, and of course a morphism of groupoids which induces a group homomorphism $\phi_* : \pi_1(L, v) \to \pi_1(D, x)$.

Claim that for each $p \in P(\Delta)$ with $x = org(p)$ and $end(p)$ in $\phi(\Lambda)$, there exists $q \in P(\Lambda)$ with $org(q) = v$ and $\phi(q) \sim p$. This will prove the surjectivity of $\phi : \pi_1(L, v) \to \pi_1(D, x)$ and hence complete the proof of the lemma. For if p is a cycle then as $\phi^{-1}(st_D(x))$ is connected there is a path r in $\phi^{-1}(st_D(x))$ from $end(q)$ to v. Then qr is a cycle with $\phi(qr) = \phi(q)\phi(r) \sim p\phi(r)$. Thus it remains to observe $\phi(r) \sim 1$ as $\phi(r)$ is a cycle in $st_D(x)$ and $st_D(x)$ is contractible.

Let $p = x_0 \cdots x_n$ and $q = y_0 \cdots y_m \in P(\Lambda)$ with $\phi(q) = x_0 \cdots x_m$, $y_0 = v$, and $m \leq n$. Pick p in $\tilde{p}$ and q so that $n - m$ is minimal subject to this constraint. If $n = m$ we are done, so assume not.

Let $u = x_{m+1}$ if $n = m+1$ and $u = x_{m+2}$ otherwise. By local connectivity there is $y \in \phi^{-1}(st_D(x_{m+1}) \cap st_D(u))$. Further if $n = m+1$ then we have $u \in \phi(\Lambda)$, so we may pick $\phi(y) = u$. Again by local connectivity there exists a path $v_0 \cdots v_k$ from y_m to y in $\phi^{-1}(st_D(x_{m+1}))$. Let $q' = (y_0 \cdots y_m) \cdot (v_0 \cdots v_k)$ and $p' = \phi(q') \cdot (\phi(v_k)x_{m+2} \cdots x_n)$. Then $\{\phi(v_i), \phi(v_{i+1})\} \in st_D(x_{m+1})$ so $x_{m+1}\phi(v_i)\phi(v_{i+1})x_{m+1} \sim 1$ for each i and hence $p \sim p'$. But $l(p') - l(q') = n - m - 1$, contrary to the choice of p and q. ∎

(2.3): *Assume $\phi : L \to D$ is locally connected and for $\Gamma = \Lambda, \Delta$, let* Comp(Γ) *be the set of connected components of* Γ. *Then*

(1) *For all $A \in$ Comp(Λ), $\phi(A)$ is contained in a unique $c(A) \in$ Comp(Δ) and the map $c :$ Comp$(\Lambda) \to$ Comp(Δ) is a bijection with inverse $B \mapsto \phi^{-1}(B)$.*

(2) *$\phi_* : H_0(L) \to H_0(D)$ is an isomorphism.*

Proof: Let $A \in$ Comp$((\Lambda)$. Then $\phi(A)$ is connected so $\phi(A)$ is contained in a unique $B \in$ Comp(Δ). Similarly for $x \in B$, $st_D(x) \subseteq B$ and $\emptyset \neq \phi^{-1}(st_D(x))$ is connected, so $\phi^{-1}(st_D(x))$ is contained in a unique $A(x) \in$ Comp(Λ).

Let $x \in \Delta$ and $y \in st_D(x)$. Then by local connectivity, $\phi^{-1}(st_D(x) \cap st_D(y)) \neq \emptyset$, so $A(x) = A(y)$. Hence as B is connected, $A(x) = A(b)$ for all $x, b \in B$, so $\phi^{-1}(B) = A$.

Therefore (1) is established and as $H_0(L)$ is free with generators $a(A) + B_0(L)$, $A \in$ Comp(Λ), where $a(A)$ is some representative for A, (1) implies (2). (Here $B_0(L)$ is the subspace of boundaries of $C_0(L)$; *cf.* section 3.) ∎

We can now establish Theorem 2. Let $\phi : L \to D$ be locally connected. By 2.2, $\phi_* : \pi_1(L, x) \to \pi_1(D, \phi(x))$ is a surjection. Further if L is connected then D is connected by 2.3. Finally if L is simply connected then by 1.11, $\pi_1(L) = 0$, so by 2.2, $\pi_1(D) = 0$, and then D is simply connected by another application of 1.11.

(2.4): *Assume θ is a 1-approximation of L by D. For $x \in \Lambda$ let $\mathcal{F}(x) = \{a \in \Delta : x \in \theta(a)\}$ regarded as a subgraph of Δ. Then*

(1) *Assume that whenever $a, b \in \Delta$ and $x \in \theta(a) \cap \theta(b)$ that there exists $y \in \theta(a) \cap \theta(b)$ such that a and b are in the same connected component*

of $\mathcal{F}(y)$ and x and y are in the same connected component of $\theta(a) \cap \theta(b)$. Then if D is simply connected, so is L.

(2) *Assume $\alpha : L \to D$ is a simplicial map such that for each $x \in \Lambda$, $st_L(x) \subseteq \theta(\alpha(x))$. Then α induces an injection $\alpha_* : \pi_1(L,x) \to \pi_1(D,\alpha(x))$. In particular if D is simply connected then so is L.*

(3) *If $\phi : L \to D$ is a simplicial map such that $\phi^{-1}(a) \subseteq \theta(a) \subseteq \phi^{-1}(st_D(a))$ for each $a \in \Delta$ and D is simply connected, then L is simply connected.*

Proof: Consider a covering $d : \hat{L} \to L$ of L and let $F = F^d$ be the local system supplied by Example 2 of section 1. For $a \in \Delta$ define $E(a) = \lim_{x \in \theta(a)} F(x)$. As $\theta(a)$ is simply connected, 1.8.2 says the map $E_{x,a} : u \mapsto \bar{u}$ is a bijection of $F(x)$ with $E(a)$ for all $x \in \theta(a)$.

Next if (a,b) is an edge in Δ then as θ is a 1- approximation there is $x \in \theta(a) \cap \theta(b)$. Define $E_{a,b} = E_{a,b,x} : E(a) \to E(b)$ by $E_{a,b} = E_{x,b} \circ E_{x,a}^{-1}$. We first observe that $E_{a,b}$ is independent of x. As $\theta(a) \cap \theta(b)$ is connected, it suffices to show $E_{a,b,x} = E_{a,b,y}$ for (x,y) an edge in $\theta(a) \cap \theta(b)$. But

(2.4.4): *If (a,b) is an edge in Δ and $x \in \theta(a) \cap \theta(b)$ then $E_{x,a}^{-1} = F_{yx} \circ E_{y,a}^{-1}$ and $E_{x,b} \circ F_{yx} = E_{y,b}$.*

So $E_{a,b,x} = E_{x,b} \circ E_{x,a}^{-1} = E_{x,b} \circ F_{yx} \circ E_{y,a}^{-1} = E_{y,b} \circ E_{y,a}^{-1} = E_{a,b,y}$, establishing the claim.

Now by definition, $E_{a,b}^{-1} = E_{b,a}$. Further if $\{a,b,c\}$ is a 2-simplex in D then by hypothesis there exists $x \in \theta(a) \cap \theta(b) \cap \theta(c)$. Then $E_{b,c} \circ E_{a,b} = E_{x,c} \circ E_{x,b}^{-1} \circ E_{x,b} \circ E_{x,a}^{-1} = E_{x,c} \circ E_{x,a}^{-1} = E_{a,c}$.

Thus E is a local system for D. We next claim:

(2.4.5): *If $q = a_0 \cdots a_n$ is a path in Δ and $p = x_0 \cdots x_n$ is a path in Λ with $x_i \in \theta(a_i) \cap \theta(a_{i+1})$ for $0 \le i < n$ and $x_n \in \theta(a_n)$, then $E_q = E_{x_n,a_n} \circ F_p \circ E_{x_0,a_0}^{-1}$.*

The proof is by induction on n. The claim is clear if $n = 0$. If $p = xy$ and $q = ab$ then $E_q = E_{ab} = E_{x,b} \circ E_{x,a}^{-1} = E_{y,b} \circ F_{xy} \circ E_{x,a}^{-1}$ by 2.4.4. Finally if $n > 1$ then $p = s \cdot xy$, $q = r \cdot ab$, and $E_q = E_{ab} \circ E_r = E_{y,b} \circ F_{xy} \circ E_{x,a}^{-1} \circ E_{x,a} \circ F_s \circ E_{x_0,a_0}^{-1} = E_{y,b} \circ F_p \circ E_{x_0,a_0}^{-1}$, completing the proof of the claim.

Finally suppose $1 \neq \tilde{p} \in \pi_1(L,v)$; then by 1.10.2, we may choose d with $F_p \neq id$. We will produce a cycle $q = c_0 \cdots c_n$ of Δ and a cycle $p' = z_0 \cdots z_n \equiv p$ such that $z_i \in \theta(c_i) \cap \theta(c_{i+1})$ for each i. Then by 2.4.5, $E_q = E_{z_0,c_0} F_{p'} E_{z_0,c_0}^{-1}$. Also as $p \equiv p'$, $F_{p'} = F_p \neq id$, so $E_q \neq id$ and hence D is not simply connected by 1.10.2

and 1.11. This will prove (1). Further in (2) we can choose $q = \alpha(p)$. Hence $\tilde{\alpha}(p) \neq 1$ in $\pi_1(D, \alpha(x))$ by 1.10.2, completing the proof of (2). Moreover under the hypotheses of (3), if $x \in \theta(a) \cap \theta(b)$ then $\phi(x) \in \{a,b\}^{\perp}$ and $x \in \theta(\phi(x))$, so $a\phi(x)b$ is a path in $\mathcal{F}(x)$, and (1) implies (3).

So it remains to produce q and p'. Let $p = x_0 \cdots x_m$. First as each 1-simplex of L is in a member of $\mathcal{F}$, there exists $a_i \in \Delta$ with $x_{i-1}, x_i \in \theta(a_i)$. Indeed under the hypothesis of (2) we can choose $a_i = \alpha(x_i)$, $p' = p$, and $q = a_0 \cdots a_m$.

So from now on assume the hypothesis of (1). Then there exists $y_i \in \theta(a_i) \cap \theta(a_{i+1})$ such that there is a path $q_i = b_{i0} \cdots b_{im(i)}$ from a_i to a_{i+1} in $\mathcal{F}(y_i)$ and a path r_i from y_i to x_i in $\theta(a_i) \cap \theta(a_{i+1})$.

Let t_i, t_i', s_i be constant paths of length $l(r_i), l(r_i), l(q_i)$ with vertex a_i, a_{i+1}, y_i, respectively. Let $q_i' = t_i \cdot q_i \cdot t_{i+1} \cdot a_{i+1}a_{i+1}$ and $p_i = r_i^{-1} \cdot s_i \cdot r_i \cdot x_i x_{i+1}$. Let $q = q_0' \cdots q_{m-1}'$ and $p' = p_0 \cdots p_{m-1}$. These paths do the job and complete the proof. ∎

Notice that 2.4.1 is Theorem 3, so Theorem 3 is established. Also we can prove analogue of Theorem 1:

(2.5): *Assume $\phi : L \to D$ is a locally simply connected simplicial map. Then for $x \in \Lambda$:*

(1) *ϕ induces an isomorphism $\phi_* : \pi_1(L,x) \to \pi_1(D, \phi(x))$ of fundamental groups.*
(2) *L is simply connected if and only if D is simply connected.*
(3) *$\phi_* : H_1(L) \to H_1(D)$ is an isomorphism if L is connected.*

Proof: By 2.2, $\phi_* : \pi_1(L,x) \to \pi_1(D,\phi(x))$ is surjective, so to prove (1) it remains to show ϕ_* is injective.

For $a \in \Delta$ define $\theta(a) = \phi^{-1}(st_D(a))$. Then as ϕ is locally simply connected, θ is a 1-approximation of L by D. Further if $x \in \Lambda$ then $st_L(x) \subseteq \theta(\phi(x))$. Hence ϕ_* is injective by 2.4.2, completing the proof of (1). Of course (1) implies (2) and as $H_1(L)$ is the abelianization of the fundamental group of L when L is connected, (1) also implies (3). ∎

3. Homology

Let K be a simplicial complex with graph Δ. Let $P = P(\Delta)$ be the path groupoid of Δ, S the closure of all 2-simplices of K, $\sim \, = \, \sim_S$, and $\Phi(K) = \tilde{P}$ the **edge path groupoid** for K. (Compare to Chapter 3, Section 6 of Spanier [6]).

Let $C_0(K)$ be the free $\mathbf{Z}$-module with basis the 0-simples Δ. An **oriented k-simplex** is an element $x_0 \wedge \cdots \wedge x_k \in \bigwedge^{k+1}(C_0(K))$ such that $\{x_0, \ldots, x_k\}$ is a k-simplex of K. Let $C_k(K)$ be the submodule of $\bigwedge^{k+1}(C_0(K))$ spanned by the oriented k-simplices.

Recall the boundary operator

$$\partial = \bigcup_i \partial_i$$

consists of the linear maps $\partial_i : C_i(K) \to C_{i-1}(K)$ where $\partial_0 = 0$ and

$$\partial_k(c) = \sum_{i=0}^{k} (-1)^i c^i$$

for each oriented k-simplex $c = x_0 \wedge \cdots \wedge x_k$, where for $J \subseteq \{0, \ldots, k\}$, $c^J = x_{i_1} \wedge \cdots \wedge x_{i_r}$ with $i_1 < \cdots < i_r$ and $\{i_1, \ldots, i_r\} = \{0, \ldots, k\} - J$. Also $Z_i(K) = ker(\partial_i)$, $B_i(K) = Im(\partial_{i+1})$, and $H_i(K) = Z_i(K)/B_i(K)$.

Define $\psi : P \to C_1(X)$ by $\psi(p) = \sum_{i=0}^{n-1} x_i \wedge x_{i+1}$ for $p = x_0 \cdots x_n$.

Define the **abelianizing relation** to be the invariant relation $\simeq$ on P generated by all paths of the form $pqp^{-1}q^{-1}$ with p and q cycles with the same origin. Then $\pi_1(K,x)/\simeq \; = \pi_1(K,x)/[\pi_1(K,x), \pi_1(K,x)]$ is the abelianization of the fundamental group $\pi_1(K,x)$.

We record some standard facts about this set up:

(3.1): *Let $p, q \in P$. Then*

(1) *$\psi : P \to C_1(K)$ is a groupoid homomorphism.*

(2) *$\psi(r^{-1}pr) = \psi(p)$ for all paths r, p with $org(p) = end(p) = org(r)$.*

(3) *ψ induces a surjective groupoid homomorphism $\psi : \Phi(K)/\simeq \to C_1(K)$.*

(4) *$p \simeq q$ if and only if p and q have the same end and origin and $\psi(p) = \psi(q)$.*

(5) *$Z_1(K)$ is generated as a $\mathbf{Z}$-module by the elements $\psi(p)$, p a cycle in Δ.*

(6) *Let $t = xyzx \in P$ with $\{x, y, z\}$ a 2-simplex of K. Then $\partial(x \wedge y \wedge z) = -\psi(t)$ and $B_1(K)$ is generated by the elements $\psi(t)$, t a 2-simplex of D.*

(7) *ψ induces an isomorphism $[\pi_1(K), \pi_1(K)] + \tilde{p} \mapsto \psi(p) + Z_1(K)$ of the abelianization of $\pi_1(K)$ with $H_1(K)$.*

Proof: See for example Exercise H in Chapter 4 of Spanier [6]. ∎

(3.2): *$H_1(K)$ is the abelianization of the fundamental group of K, so if K is simply connected then $H_1(K) = 0$.*

Proof: This is just a restatement of 3.1.7. ∎

Define the **link** of a simplex s of K to be the subcomplex $Link_K(s)$ of $st_K(s)$ consisting of all simplices t with $t \cap s = \emptyset$.

(3.3): *Let $x \in \Delta$ and assume $y, z \in \Delta$ are in different connected components of $Link_K(x)$. Then*

(1) *If $p = x_0 \cdots x_n$ is a cycle in Δ with $x_0 = y$, $x_1 = x$, $x_2 = z$, and $x_i \neq x$ for $i \neq 1$, then $H_1(K) \neq 0$.*
(2) *If $H_1(K) = 0$ then $\{x\} = y^\perp \cap z^\perp$.*

Proof: Assume $H_1(K) = 0$. Then in (1), $\psi(p) = \partial(\sum_{i \in I} \sigma_i)$ for some family $(\sigma_i : i \in I)$ of oriented 2-simplices by 3.1.5 and 3.1.6. Let $\bar{C}_k(K) = C_k(K)/2C_k(K)$, so that $\bar{C}_k(K)$ is a $GF(2)$-space. Let $\bar{\psi} = \phi \circ \psi$, where $\phi : C \to \bar{C}$ is reduction modulo 2. Then

(3.3.3): $\bar{\psi}(p) = \sum_{i \in I} \bar{\psi}(t_i)$

where t_i is the triangle in Δ with $\partial \sigma_i = \psi(t_i)$; *cf.* 3.1.6.

Let L be the connected component of $Link(x)$ containing y and J consist of those $j \in I$ such that t_i contains an edge xu with $u \in L$. Then

(3.3.4): $\sum_{j \in J} \bar{\psi}(t_j) = \sum_{j \in J} \bar{\psi}(u_j v_j) + \bar{\psi}(x u_j) + \bar{\psi}(x v_j)$

where $t_j = x u_j v_j x$. Now as $x_i \neq x$ for $i \neq 1$ and $z \notin L$, 3.3.3 and 3.3.4 imply

(3.3.5): $\sum_{j \in J} \bar{\psi}(x u_j) + \bar{\psi}(x v_j) = \bar{\psi}(xy)$

This is impossible as the sum on the left in 3.3.5 has an even number of terms, whereas 3.3.5 says y is the unique member of L such that $\bar{\psi}(xy)$ appears an odd number of times as a term.

So (1) is established. Now if $u \in y^\perp \cap z^\perp - \{x\}$ then $p = yxzuy$ is a cycle satisfying the hypothesis of (1), so (1) implies (2).

Finally we record the following standard fact:

(3.4): *Let L be a n-dimensional subcomplex of K such that*

(1) *$H_n(L) \neq 0$, and*
(2) *Each n-simplex of L is a maximal simplex of K.*

Then $H_n(K) \neq 0$.

Proof: If $0 \neq z \in Z_n(L)$, then $z \in Z_n(K)$ and hypothesis (2) implies that $z \notin B_n(K)$, hence $z + B_n(K)$ is a nontrivial element of $H_n(K)$. ∎

4. Carriers

In this section D is a simplicial complex with graph Δ.

Following Walker in [7] and Bjõrner in [3], define an n-**connected carrier** from D to a topological space T to be a function $C : D \to 2^T$ from the simplices of D into the power set of T such that

(C1) If $s \subseteq t$ are simplices of D then $C(s) \subseteq C(t)$.

(C2) $C(s)$ is $min(n, dim(s))$-connected for all simplices s of D.

We say C is a **contractible carrier** if C is n-connected for all n. If $f : |D| \to T$ is continuous then we say C *carries* f if $f(|s|) \subseteq C(s)$ for all simplices s of D. For $g : |D| \to T$ continuous we write $f \simeq g$ to indicate that f and g are homotopic.

(4.1): *Let C be an n-connected carrier from D to a topological space T. Then*

(1) *If $f, g : |D^n| \to T$ are carried by C then $f \simeq g$.*

(2) *Each function $f : \Delta \to T$ with $f(x) \in C(x)$ for all $x \in \Delta$ extends to a continuous map $f : |D^{n+1}| \to T$ carried by C.*

Proof: This is Lemma 10.1 in [3], which refers the reader elsewhere for a proof. A proof of a somewhat weaker result appears in Lemma 2.1 of [7], but this proof works in our situation too. For completeness we repeat it here.

The proof is by induction on n. When $n = -1$ part (1) is vacuously satisfied as $D^{-1} = \emptyset$, while in (2) as D^0 consists of the vertices of Δ but no edges, the lemma is trivial. Thus the induction is anchored and we may take $n \geq 0$. Further proceeding by induction on n, in (2) we may assume f is defined on $|D^n|$ so as to be carried by the restriction of C to D^n. In (1) define $H : |D^n| \times \{0,1\} \to T$ by $H(a,0) = f(a)$ and $H(a,1) = g(a)$. We may assume H is extended continuously to $H : |D^{n-1}| \times I \to T$, where I is the unit interval.

Next in (2), if we can extend f continuously to $|s|$ for all $(n+1)$-simplices s then we have our continuous extension to $|D^{n+1}|$. Similarly in (1) it suffices to extend H to $|s| \times I$ for all n-simplices s. Let $B(s)$ be the union of the faces $|t|$, as t ranges over all $dim(s)-1$ subsimplices of s. Then in (2), we have a homeomorphism $|s| \cong E^{n+1}$ mapping $B(s)$ to S^n, so as $C(s)$ is n-connected, there exists a continuous extension of f to $|s|$, completing the proof of (2). Finally in (1), there is a homeomorphism of $|s| \times I$ with E^{n+1} mapping $(B(s) \times I) \cup (|s| \times \{0,1\})$ to S^n, so we can extend H continuously, and (1) is established. ∎

(4.2): *Let $\phi : L \to D$ be a simplicial map and assume Ξ is a map from simplices of D to n-connected subcomplexes of L, Φ is a map from k-simplices of D to $min(n,k)$-connected subcomplexes of D, (for all k) such that $\Xi(s_1) \subseteq \Xi(s_2)$ and $\Phi(s_1) \subseteq \Phi(s_2)$ whenever $s_1 \subseteq s_2$, and that*

(a) *$t \subseteq \Xi(\phi(t))$ for each simplex t of L, and*

(b) *$\phi(\Xi(s)) \cup s \subseteq \Phi(s)$ for each simplex s of D.*

Then

(1) *$\phi : L^n \to D^n$ is a homotopy equivalence.*

(2) *If $n \neq 1$ then L is n-connected if and only if D is n-connected.*

(3) *If $n = 1$ and $\Phi(s)$ is 2-connected for each 2-simplex s of D then $\phi_* : \pi_1(L,x) \to \pi_1(D,\phi(x))$ is a surjection, so if L is simply connected, so is D.*

(4) *If $n = 1$ and $\Xi : a \mapsto \Xi(a)$ is a 1-approximation of L by D with $st_L(x) \subseteq \Xi(\phi(x))$ for all vertices x of L then $\phi_* : \pi_1(L,x) \to \pi_1(D,\phi(x))$ is injective, so if D is simply connected, so is L.*

(5) *If $n = 1$ and $\Xi : a \mapsto \Xi(a)$ is a 1-approximation of L by D with $\Xi(a) \subseteq \phi^{-1}(st_D(a))$ for each $a \in \Delta$ and D is simply connected then so is L.*

(6) *If $\Xi(s), \Phi(s)$ are contractible for each simplex s of D then $\phi : L \to D$ is a homotopy equivalence.*

Proof: Proofs of some parts of the lemma are sketched in [3] as suggested by [7]; for completeness we fill in a few details of the proof. Without loss $dim(D) \leq n+1 \geq dim(L)$. For s a simplex of D define $C(s) = |\Xi(s)|$ and $F(s) = |\Phi(s)|$ and for t a simplex of L let $E(t) = C(\phi(t))$. Then by hypothesis, C, E, and F are n-connected carriers. Let $f = |\phi| : |L| \to |D|$. By 4.1.2, there exists $g : |D| \to |L|$ carried by C.

Now $(g \circ f)(|t|) \subseteq C(\phi(t)) = E(t)$, so E carries $g \circ f$. Also by (a), $|t| \subseteq E(t)$, so E carries $id_{|L|}$, and hence by 4.1.1, $g \circ f \simeq id_{|L^n|}$.

Similarly $(f \circ g)(|s|) \subseteq f(C(s)) = f(|\Xi(s)|) = |\phi(\Xi(s))| \subseteq F(s)$ and $|s| \subset F(s)$ by two applications of (b). So $f \circ g \simeq id_{|D^n|}$ by 4.1.1. Therefore $\phi : L^n \to D^n$ is a homotopy equivalence, and of course under the hypotheses of (6), $\phi : L \to D$ is a homotopy equivalence. So (1) and (6) are established.

Suppose $n = 1$. Under the hypotheses of (3), F is an $n+1$-connected carrier, so $f \circ g \simeq id_{|L|}$ by 4.1.1 and hence $f_* \circ g_* = id$, where $f_* : \pi_1(|L|) \to \pi_1(|D|)$ and

$g_* : \pi_1(|D|) \to \pi_1(|L|)$.

Therefore (3) holds. Similarly 2.4.2 and 2.4.3 imply (4) and (5), respectively.

Thus it remains to show that $H_n(L) = 0$ if and only if $H_n(D) = 0$. To do so we adopt the notation in Chapter 4 of [6] and observe that it suffices to show the induced maps $\Delta(f) : Z_n(|L^n|) \to Z_n(|D^n|)$ and $\Delta(g) : Z_n(|D^n|) \to Z_n(|L^n|)$ of page 161 of [6] are inverses of each other. For then if $H_n(L) = 0$ then $Z_n(|L|) = B_n(|L|)$, so $Z_n(|D|) = \Delta(f)(Z_n(|L|)) = \Delta(f)(B_n(|L|)) \subseteq B_n(|D|)$, and hence $H_n(D) = 0$. Similarly if $H_n(D) = 0$ then $H_n(L) = 0$.

Now as $g \circ f \simeq id_{|L^n|}$, $\Delta(g \circ f)_n - id_{C_n(|L|)} = \partial_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial_n$ for some chain homotopy σ of degree 1 on the chain group $C(|L^n|)$. (*cf.* 4.4.4 in [6]) But $dim(L^n) = n$, so $\partial_{n+1} \circ \sigma_n = 0$, while $\sigma_{n-1} \circ \partial_n = 0$ on $Z_n(|L|)$, so $\Delta(g \circ f)$ is the identity on $Z_n(|L|)$, completing the proof. ∎

Recall the **order complex** $\mathcal{O}(X)$ of a poset X is the simplicial complex on X whose simplices are the finite chains in X. We often abuse notation and write X for $\mathcal{O}(X)$. Notice that if $\phi : X \to Y$ is an order preserving map then ϕ defines a simplicial map $\phi : \mathcal{O}(X) \to \mathcal{O}(Y)$. For $x \in X$, write $X(\geq x)$ for $\{y \in X : y \geq x\}$.

(4.3): *(Quillen) Let L, D be the order complexes of posets P, Q, respectively and $\phi : P \to Q$ an order preserving map. Then*

(1) *If $\phi^{-1}(Q(\geq q))$ is n-connected for all $q \in Q$ then $\phi : L^n \to D^n$ is a homotopy equivalence and L is n-connected if and only if D is n-connected.*

(2) *If $\phi^{-1}(Q(\geq q))$ is contractible for all $q \in Q$ then $\phi : L \to D$ is a homotopy equivalence.*

Proof: Quillen gives a proof in Prop. 1.6 and 7.6 of [5]. The proof of (2) given here comes from [7]; the proof of (1) is suggested in [3], although simple connectivity does not appear to be addressed when $n = 1$. In any event we supply the proof for completeness.

We first produce maps Ξ and Φ as in 4.2. For s a simplex of D define $\Xi(s) = \phi^{-1}(Q(\geq min(s)))$ and $\Phi(s) = Q(\geq min(s))$, where $min(s)$ is the minimum element of the chain s. Then by hypothesis, $\Xi(s)$ is n-connected, while $\Phi(s)$ is conically contractible. Visibly hypotheses (a) and (b) of 4.2 are satisfied, so 4.2 completes the proof. Notice $\Xi : a \mapsto \Xi(a)$ is a 1-approximation of L by D with $\Xi(a) \subseteq \phi^{-1}(st_D(a))$ for each $a \in Q$, so 4.2.5 applies when $n = 1$. ∎

Define a cover $\mathcal{F}$ of D to be *contractible* if all finite intersections are contractible or empty.

(4.4):

(1) *If $\mathcal{F}$ is a contractible cover of D then D and the nerve $N(\mathcal{F})$ have the same homotopy type.*

(2) *Let $I = \{1, \cdots, m\}$ and $\mathcal{F} = \{F(i) : i \in I\}$ a cover of D such that for all $1 \leq k \leq n+2$ and all k-subsets J of I, $F(J) = \bigcap_{j \in J} F(j)$ is $(n-k+1)$-connected. Then D is n-connected.*

Proof: This is well known and essentially Theorem 2.1 in Folkman [4]. The proof we give for (1) appears in 10.6 of [3], while the proof of (2) is essentially Folkman's proof. Let $N = N(\mathcal{F})$. Write $sd(K)$ for the barycentric subdivision of a simplicial complex K; *ie.* the order complex of the poset of simplices of K. First the proof of (1). Define $f : P = sd(D) \to sd(N) = Q$ by $f(s) = \{F \in \mathcal{F} : s \subseteq F\}$. Then $s \subseteq \bigcap_{F \in f(s)} F$, so $f(s)$ is a simplex of N. Also if $t \subseteq s$ then $f(s) \subseteq f(t)$, so f is an order preserving map from P^* to Q, where P^* is the dual poset to P. (Notice $\mathcal{O}(P) = \mathcal{O}(P^*)$.)

Claim $f^{-1}(Q(\geq q))$ is contractible for all $q \in Q$; if so 4.3.2 will complete the proof of (1). Define $\bigcap_{F \in q} F = F_q$. As q is a simplex of N, $F_q \neq \emptyset$, so F_q is contractible by hypothesis. Further $f^{-1}(Q(\geq q))$ consists of all simplices s with $f(s) \geq q$; that is with $q \subseteq f(s)$ or better with $s \subseteq F_q$. So $f^{-1}(Q(\geq q)) = F_q$ is contractible, completing the proof of (1).

The proof of (2) is by induction on m. Let $I' = I - \{m\}$, $\mathcal{F}' = \{F(i) : i \in I'\}$, and $D' = \bigcup_{F \in \mathcal{F}'} F$. Then by induction on m, D' is n-connected, while by hypothesis, $F(m)$ is n-connected. Let $E(i) = F(\{i, m\})$ for $i \in I'$, and $\mathcal{E} = \{E(i) : i \in I'\}$. Then $\mathcal{E}$ is a cover of $F(m) \cap D'$ satisfying our hypotheses for $m-1$, $n-1$, so by induction on m, $F(m) \cap D'$ is $(n-1)$-connected.

Finally we have the following exact subsequence of the Mayer-Vietoris sequence:

$$\tilde{H}_k(D') \oplus \tilde{H}_k(F(m)) \to \tilde{H}_k(D) \to \tilde{H}_{k-1}(F(m) \cap D')$$

So as $\tilde{H}_k(D') = \tilde{H}_k(F(m)) = \tilde{H}_k(F(m) \cap D') = 0$ for $k < n$ and $\tilde{H}_n(D') = \tilde{H}_n(F(m)) = 0$, we conclude $\tilde{H}_k(D) = 0$ for $k \leq n$.

Thus it remains to observe that D is simply connected when $n \geq 1$. But this follows from Theorem 3 as I is contractible when regarded as the complex of all nonempty subsets of I and $i \mapsto F(i)$ is a 1-approximation of D by I. ∎

5. The proof of Theorem 1

In this section D is a simplicial complex with graph Δ.

(5.1): *Assume s is a simplex of the clique complex D such that $\mathcal{F} = \{st_D(x) : x \in s\}$ is a cover of D. Then D is contractible.*

Proof: Suppose first $s = \{x\}$ is 0-dimensional. Then for each simplex t, $t \cup \{x\}$ is a simplex. Thus the map $d : D \to D$ defined by $d(y) = x$ for all $y \in \Delta$ is contiguous to the identity, so D is contractible.

Now the general case. As D is a clique complex, 2.1 says that for $t \subseteq s$, $st_D(t) = \bigcap_{x \in t} st_D(x)$. Also $st_D(t)$ is contractible by the previous paragraph as $st_D(t) = st_{st_D(t)}(x)$ for $x \in t$. So 4.4.2 completes the proof. ∎

(5.2): *Let θ be an n-approximation of a simplicial complex L by D. Then*

(1) *$\Xi(s) = \bigcup_{a \in s} \theta(a)$ is n-connected, while if D is a clique complex then $\Phi(s) = \bigcup_{a \in s} st_D(a)$ is contractible for each simplex s of D.*

(2) *If $\phi : L \to D$ is a simplicial map with $\phi^{-1}(a) \subseteq \theta(a)$ for each $a \in \Delta$, then $t \subseteq \Xi(\phi(t))$ for each simplex t of L and if $n \neq 1$ and D is n-connected, then L is n-connected.*

(3) *If $\phi : L \to D$ is a simplicial map with $\phi^{-1}(a) \subseteq \theta(a) \subseteq \phi^{-1}(st_D(a))$ for each $a \in \Delta$ then $\phi(\Xi(s)) \cup s \subseteq \Phi(s)$ for each simplex s of D, and if in addition D is a clique complex then D is n-connected if and only if L is n-connected.*

Proof: Part (1) follows from 4.4.2 and 5.1. So assume $\phi : L \to D$ is a simplicial map with $\phi^{-1}(a) \subseteq \theta(a)$ for each $a \in \Delta$. Then visibly $t \subset \Xi(\phi(t))$ for each simplex t of L. Further if D is n-connected, let $\Phi^*(s) = D$ for each simplex s of D. Then Φ^* serves in the role of the map Φ of 4.2, so that 4.2.2 and (1) imply (2).

Finally assume also $\theta(a) \subseteq \phi^{-1}(st_D(a))$ for each $a \in \Delta$. Then visibly $\phi(\Xi(s)) \cup s \subseteq \Phi(s)$. Hence 4.2 and (1) complete the proof of (3).

We close this section with the proof of Theorem 1. So assume $\phi : L \to D$ is a locally n-connected simplicial map and D is a clique complex. For $a \in \Delta$, let $\theta(a) = \phi^{-1}(st_D(a))$. Then as ϕ is locally n-connected, θ is an n-approximation of L by D. Moreover the hypotheses of 5.2.3 are satisfied, so by 5.2.3, L is n-connected if and only if D is n-connected. Further by 5.2.2 and 5.2.3, the maps Ξ and Φ of 5.2.1 satisfy the hypotheses of 4.2. So by 4.2.1, $\phi : L^n \to D^n$ is a homotopy equivalence. In particular if $n > 1$ then $\phi_* : \pi_1(L, x) \to \pi_1(D, \phi(x))$ is an isomorphism. Similarly if ϕ is locally contractible then by 4.2.6, $\phi : L \to D$ is a homotopy equivalence.

Finally assume $n = 1$. Then by 4.2.3 and 4.2.4, $\phi_* : \pi_1(L, x) \to \pi_1(D, \phi(x))$ is an isomorphism. Therefore the proof of Theorem 1 is complete.

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